Benefits of Angular Expression to Reconstruction Algorithms for Collimators with Spatially Varying Focal Lengths

Jiangsheng You,* Shanglian Bao, and Zhengrong Liang

***Abstract*— Fan-beam collimators with spatially varying focal lengths are used in single photon emission computed tomography (SPECT) to improve the imaging sensitivity and to reduce the reconstruction artifacts resulting from the truncation of projection data. An angular representation of the detector coordinates for the projection data is adopted to investigate several aspects of the reconstruction problem for this type of collimation. A rebinning reconstruction algorithm is derived. We prove a conjecture posed by Zeng and Gullberg and obtain a simplified form of their backprojection filtering algorithm. Some computer simulations are presented to investigate the performance of the rebinning algorithm.**

***Index Terms*—Angular detector coordinate, backprojection filtering, collimators with spatially varying focal length, convolution backprojection, fan-beam geometry.**

I. Introduction

In single photon emission computed tomography (SPECT), a fan-beam collimator with spatially varying focal lengths is used with a gamma camera to improve the imaging sensitivity and to reduce the reconstruction artifacts resulting from the truncation of projection data. The focal lengths increase from a minimum at the center to a maximum at the edge of the object. Thus, those projection rays with short focal lengths have good sensitivity in the central region of interest and others with long focal lengths will greatly reduce truncation artifacts near the edge of the object.

Some convolution backprojection algorithms have been developed for parallel-beam geometry [1] and later extended to various fan-beam geometries [2]–[4]. However, no convolution backprojection algorithm has been derived for spatially varying focal length fan-beam geometry. Zeng and Gullberg proposed a summed convolution backprojection algorithm to reconstruct the image approximately [5]. Cao and Tsui applied the double integral to address the reconstruction problem for this type of geometry [6]. Recently, Zeng and Gullberg implemented a backprojection filtering algorithm to reconstruct the image more accurately [7]. In their work, a conjecture was stated that a convolution backprojection algorithm does not exist for collimators with spatially varying focal lengths.

In this paper, based on an angular representation of the detector coordinate for the projection data, we derived a rebinning reconstruction algorithm for spatially varying focal length fan-beam geometry. Moreover, we proved the conjecture of Zeng and Gullberg and obtained a simplified version of the backprojection filtering algorithm as proposed in [7].

II. Collimator Geometry

The collimator geometry discussed in this paper is shown in Fig. 1. Assume that the flat detector surface is parallel to y-axis with a distance R away from the origin. The detected value at a point A on the detector is the integral of image function $f(x, y)$ on the line SA, where S is located on the x-axis. Let α denote the angle between SA and x-axis. The focal point S varies away from the origin as a function $D(\alpha)$, which can also be expressed as a function $D(s)$, where s is the distance between A and the central ray of the detector. When the focal length increases from a minimum at the center to a maximum at the edge of the object, $D(\alpha)$ or $D(s)$ will increase with respect to $|\alpha|$ or $|s|$, respectively. Obviously, each projection ray in Fig. 1 can be determined uniquely by the departure angle α or the distance s. In this paper, we assume that $D(-\alpha) = D(\alpha)$ and $D(-s) = D(s)$.

Rotate the detector around the origin with an angle Φ, see Fig. 2. Every point on the detector surface can be uniquely determined by an angular coordinate (Φ, α) or a flat coordinate (Φ, s). Comparing these two coordinates with the corresponding coordinate (θ, l) of parallel-beam geometry, we have

$$l = \frac{sD(s)}{\sqrt{[D(s)+R]^2 + s^2}} \tag{1}$$

$$\theta = \Phi + \frac{\pi}{2} + \tan^{-1}\left[\frac{s}{D(s)+R}\right] \tag{2}$$

Manuscript received February 7, 1996; revised July 15, 1997. This work was supported by NNSF of China under Grants 39570223 and 19675005, in part by National Institutes of Health (NIH) under Grants HL51466 and NS35853, and by the American Heart Association under EI Awards. The Associate Editor responsible for coordinating the review of this paper and recommending its publication was G. T. Gullberg. *Asterisk indicates corresponding author.*

*J. You was with Institute of Heavy Physics, Peking University, Beijing 100871 PR China. He is now with the Department of Radiology, the State University of New York (SUNY), Stony Brook, NY 11794 USA (e-mail: jshyou@clio.rad.sunysb.edu).

S. Bao is with Institute of Heavy Physics, Peking University, Beijing 100871 PR China.

Z. Liang is with the Departments of Radiology and Computer Science, and the Program in Biomedical Engineering, the State University of New York (SUNY), Stony Brook, NY 11794 USA.

Publisher Item Identifier S 0278-0062(97)07891-9.

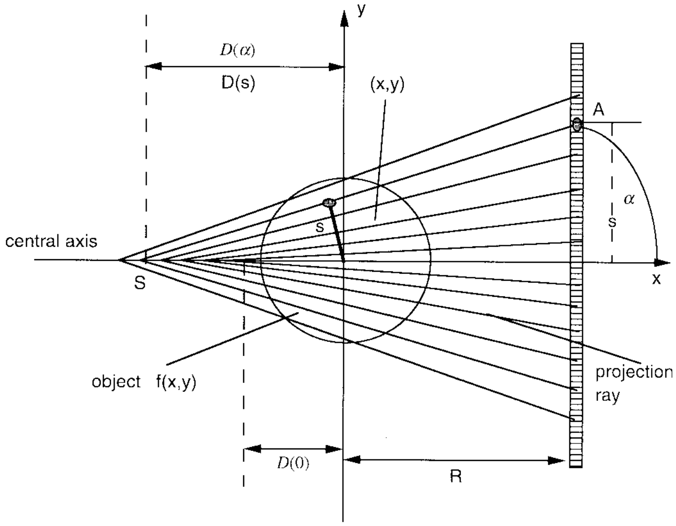

Fig. 1. Spatially varying focal length fan-beam geometry.

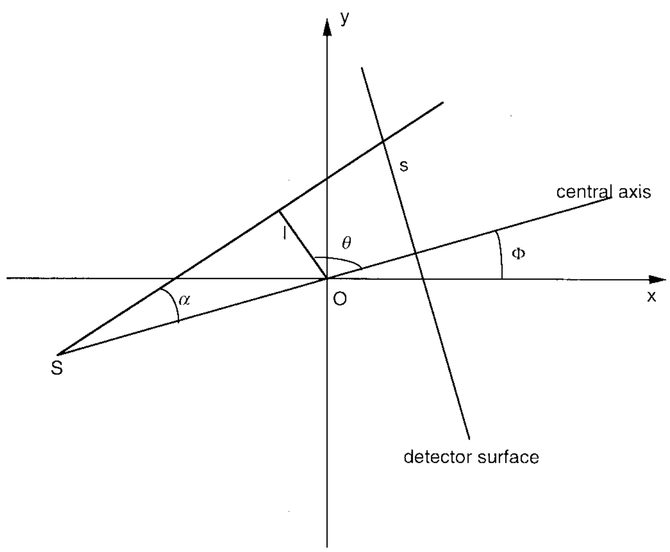

Fig. 2. No caption was provided.

or

$$l = D(\alpha)\sin\alpha \tag{3}$$

$$\theta = \Phi + \frac{\pi}{2} + \alpha. \tag{4}$$

The Jacobian from $ds\,d\Phi$ to $dl\,d\theta$ is

$$J(s) = \frac{D(s)[D(s)+R]^2 + s^3 D'(s) + sR[D(s)+R]D'(s)}{\{[D(s)+R]^2 + s^2\}^{3/2}} \tag{5}$$

and the Jacobian from $d\alpha\,d\Phi$ to $dl\,d\theta$ is

$$J(\alpha) = D'(\alpha)\sin\alpha + D(\alpha)\cos\alpha \tag{6}$$

where $D'(\alpha)$ is the differentiation of $D(\alpha)$ with respect to the variable α. Let $R_0 = \min\{D(0), R\}$ and $B(R_0)$ denote the disk with a radius R_0 and the center at the origin, then, $J(\alpha)$ and $J(s)$ are regular in $B(R_0)$. Hereafter, we will assume that the support of image function $f(x, y)$ is contained in $B(R_0)$.

Zeng and Gullberg [5], [7], Cao and Tsui [6] derived some reconstruction algorithms of spatially varying focal length fan-beam geometry based on the flat coordinate representation $D(s)$. We will use the angular coordinate representation $D(\alpha)$ to investigate the reconstruction problem.

III. A Rebinning Reconstruction Algorithm

In Section II, we showed that every point on the detector can be uniquely determined by a pair of parameters (Φ, α). Let $P_\Phi(\alpha)$ denote the detected value at certain point on the detector surface and $p(l, \theta)$ denote its corresponding expression of parallel-beam geometry.

Assume that the subtending angle of $P_\Phi(\alpha)$ is $2\alpha_0$, i.e., $\alpha \in [-\alpha_0, \alpha_0]$. To obtain complete projections for spatially varying focal length fan-beam geometry, Φ needs to rotate from $-\pi/2 - \alpha_0$ to $\pi/2 + \alpha_0$. However, for convenience, we will assume that $\Phi \in [0, 2\pi]$ and $\alpha \in [-\pi/2, \pi/2]$.

The radon inversion formula can be written as

$$f(x, y) = \tfrac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} |\omega| P(\omega, \theta) \cdot \exp[2\pi i\omega(x\cos\theta + y\sin\theta)]\,d\omega\,d\theta \tag{7}$$

where $P(\omega, \theta)$ denotes the Fourier transform (FT) of $p(l, \theta)$ with respect to the variable l. Define an approximate filter $W(\omega)$

$$W(\omega) = e^{-2\pi\epsilon|\omega|}\,|\omega| \tag{8}$$

where ε is a small constant, and calculate its inverse FT

$$H(t) = \frac{\epsilon^2 - t^2}{2\pi^2(\epsilon^2 + t^2)^2} \tag{9}$$

we get

$$f(x, y) \approx \tfrac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} p(l, \theta) \cdot H(x\cos\theta + y\sin\theta - l)\,dl\,d\theta. \tag{10}$$

Let (r, φ) denote the polar coordinate expression of (x, y). By coordinate transforms (3) and (4), (10) can be rewritten as

$$f(r, \varphi) \approx \tfrac{1}{2}\int_0^{2\pi} d\theta \int_{-(\pi/2)}^{\pi/2} P_{\theta-\alpha-\pi/2}(\alpha) \cdot H[s - D(\alpha)\sin\alpha]J(\alpha)\,d\alpha|_{s=r\cos(\theta-\varphi)}. \tag{11}$$

At first glance, (11) is very similar to the convolution backprojection procedure, but it is not, as shown later, a convolution backprojection formulation. The advantage of (11) is that, as compared to the algorithms [5]–[7], the rebinning procedure can be easily implemented without extra rebinning interpolation if the grid intervals of α and Φ are the same. Since we can not use fast Fourier transform (FFT) to compute (11), the reconstruction may need more computation time. Performance of this rebinning reconstruction algorithm will be studied by computer simulations later.

IV. Proof of the Conjecture of Zeng and Gullberg

Now we turn to the conjecture of [7] which states that a direct convolution backprojection algorithm does not exist for spatially varying focal length fan-beam geometry. Let $\epsilon \to 0$ in (11), we have

$$f(r, \varphi) = \frac{1}{2} \lim_{\epsilon \to 0} \int_0^{2\pi} d\Phi \int_{-(\pi/2)}^{\pi/2} P_\Phi(\alpha) H[s - D(\alpha) \sin \alpha] \cdot J(\alpha)\, d\alpha|_{s = r \cos(\Phi + \alpha + \pi/2 - \varphi)}. \tag{12}$$

Define $\beta = \Phi - \varphi$, then $s - D(\alpha) \sin \alpha = -[r \sin \alpha \cos \beta + r \cos \alpha \sin \beta + D(\alpha) \sin \alpha]$.

To prove the conjecture, we first assume that a direct convolution backprojection algorithm does exist, then we will deduce a contradiction. Here, it needs to point out that, in essence, the existence of a direct convolution backprojection algorithm means that $s - D(\alpha) \sin \alpha$ can be expressed as a variable-separated form. For example, $s - D(\alpha) \sin \alpha = r \cos \beta - l$ exists for the parallel-beam geometry in Cartesian coordinate, and $s - D(\alpha) \sin \alpha = -[r \sin \beta / (D + r \cos \beta) + \tan \alpha] / [(D + r \cos \beta) \cos \alpha]$ holds for the standard fan-beam geometry in the angular coordinate. Therefore, the existence of a direct convolution backprojection procedure requires that the double integral (12) can be expressed as a separately serial integral. Then, if a direct convolution backprojection algorithm exists for spatially varying focal length fan-beam geometry, we should have the following expression:

$$s - D(\alpha) \sin \alpha = f_1(\alpha) g_1(\beta) [g_2(\beta) - f_2(\alpha)] \tag{13}$$

where $f_1(\alpha)$ and $f_2(\alpha)$ are functions of α and $g_1(\beta)$ and $g_2(\beta)$ are functions of β. To some extent, this expression partially comes from the idea of summed convolution backprojection algorithm [5]. Since s is 2π periodic with respect to β, $g_1(\beta)$, and $g_2(\beta)$ can be expanded as the Fourier series

$$g_1(\beta) = \sum_{k=0}^{\infty} (a_k \cos k\beta + b_k \sin k\beta) \tag{14}$$

$$g_2(\beta) = \sum_{k=0}^{\infty} (c_k \cos k\beta + d_k \sin k\beta). \tag{15}$$

Compare the coefficients in both sides of (13), we have

$$a_0 f_1(\alpha)[f_2(\alpha) - c_0] = D(\alpha) \sin \alpha \tag{16}$$

$$f_1(\alpha)\{a_1[f_2(\alpha) - c_0] - a_0 c_1\} = \sin \alpha \tag{17}$$

$$f_1(\alpha)\{b_1[f_2(\alpha) - c_0] - a_0 d_1\} = \cos \alpha \tag{18}$$

and

$$(a_1 d_1 - b_1 c_1) D(\alpha) \sin \alpha = a_0 d_1 \sin \alpha - a_0 c_1 \cos \alpha. \tag{19}$$

Note that (19) holds for all $\alpha \in [-\pi/2, \pi/2]$. When $a_1 d_1 - b_1 c_1 \neq 0$, we have $(a_1 d_1 - b_1 c_1) D(\alpha) = a_0 d_1$, then $D(\alpha) = constant$, i.e., fan-beam geometry with spatially varying focal lengths becomes the standard fan-beam geometry. When $a_1 d_1 - b_1 c_1 = 0$, we have $a_0 = 0$ or $d_1 = c_1 = 0$. If $a_0 = 0$, from (16) we get $D(\alpha) = 0$, this yields a contradiction; if $d_1 = c_1 = 0$, this contradicts to (17) and (18). These arguments indicate that (13) holds only for $D(\alpha) = constant$. Therefore, we proved the conjecture of [7].

V. A Simplification of the Backprojection Filtering Algorithm

Based on the angular representation above, we now consider the backprojection filtering algorithm [7] which was derived by using the flat coordinate (Φ, s). The use of the flat coordinate (Φ, s) leads to complicated computations in the rebinning procedure. As the calculations of [7], it needs to solve an implicit equation

$$\hat{s} = \frac{[D(\hat{s}) + R]\sqrt{x^2 + y^2} \sin(\beta + \alpha)}{D(\hat{s}) - \sqrt{x^2 + y^2} \cos(\beta + \alpha)} \tag{20}$$

for every fixed α and β, where $D(\hat{s})$ is the focal length function. By the angular coordinate (Φ, α), we found that the computational complications can be greatly reduced. Let

$$b(x, y) = \frac{1}{2} \int_0^{2\pi} p(x \cos \theta + y \sin \theta, \theta)\, d\theta \tag{21}$$

where $p(x \cos \theta + y \sin \theta, \theta)$ is the radon transform of the image function $f(x, y)$ for parallel-beam geometry. From discussions about the Rho-filtered layergram reconstruction algorithm in [8], we have

$$b(x, y) = f(x, y) * \frac{1}{r} \tag{22}$$

where $*$ denotes the two-dimensional convolution and $r = \sqrt{x^2 + y^2}$. If (r, φ) denotes the polar coordinate representation of (x, y), then (21) becomes

$$b(x, y) = \frac{1}{2} \int_0^{2\pi} p[r \cos(\varphi - \theta), \theta]\, d\theta. \tag{23}$$

By coordinate transforms (3) and (4), there exists a unique pair (Φ, α) satisfying

$$P_\Phi(\alpha) = p[r \cos(\varphi - \theta), \theta] \tag{24}$$

and

$$\begin{cases} r \cos(\varphi - \theta) = D(\alpha) \sin \alpha \\ \theta = \Phi + \dfrac{\pi}{2} + \alpha. \end{cases} \tag{25}$$

Rearranging the terms in (25) we get

$$\begin{cases} \theta = \varphi + \arccos\left[\dfrac{D(\alpha) \sin \alpha}{r}\right] \\ \Phi = \varphi - \alpha - \dfrac{\pi}{2} + \arccos\left[\dfrac{D(\alpha) \sin \alpha}{r}\right]. \end{cases} \tag{26}$$

Therefore

$$b(x, y) = \frac{1}{2} \int_{-\alpha(r)}^{\alpha(r)} P_{\varphi - \alpha - \pi/2 + \arccos[D(\alpha) \sin \alpha / r]} \cdot (\alpha) d \arccos\left[\frac{D(\alpha) \sin \alpha}{r}\right] \tag{27}$$

where the integral limits $\alpha(r)$ is given by $D(\alpha) \sin \alpha = r$. Let $B(u, v)$ be the FT of $b(x, y)$, it is well-known that the inverse FT of $\sqrt{u^2 + v^2} B(u, v)$ is exactly the image function $f(x, y)$. Therefore, (27) offers an explicit form and can be easily implemented.

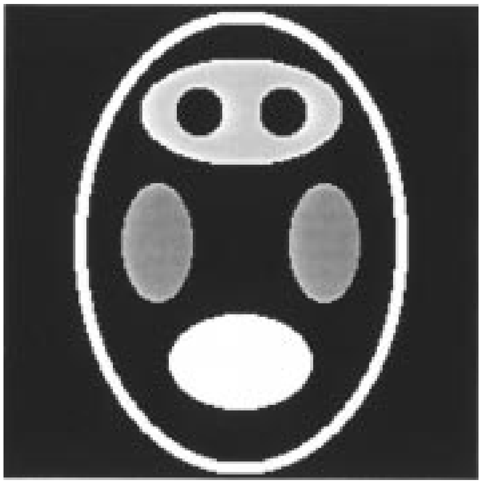

Fig. 3. Computer-generated phantom.

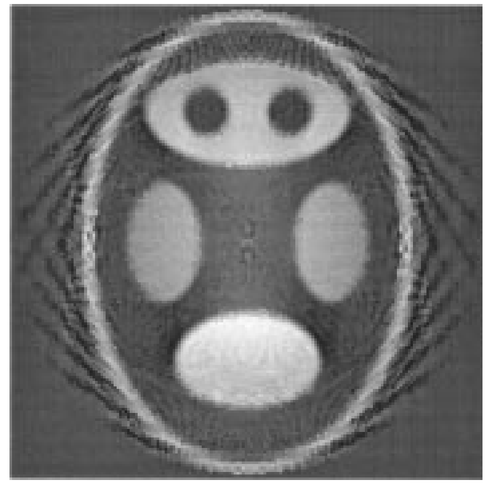

Fig. 4. Reconstructed image for focal length function 1).

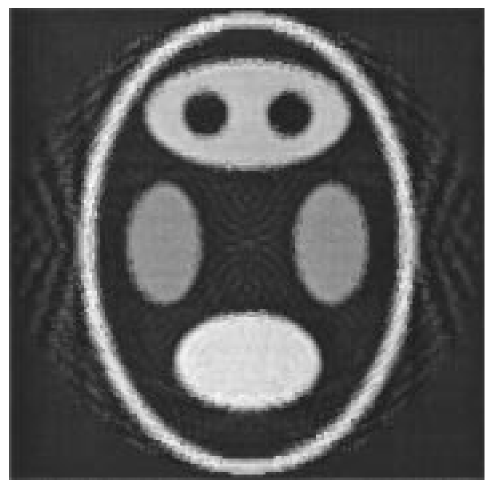

Fig. 5. Reconstructed image for focal length function 2).

The numerical implementation of (27) is the same as the procedure of [7]. The only difference is that (27) computes the backprojection directly without solving the implicit equation (20). Therefore, we omit the numerical simulations for the implementation of (27).

VI. Computer Simulations

In order to test the rebinning reconstruction algorithm (11), several experiments have been performed. In these experiments, we used a number of computer-generated projection data sets of a mathematically defined but medically realistic phantom.

The phantom consists of six superimposed ellipses and two disks. A 200×200 digitization of the phantom is shown in Fig. 3. The image function is defined on a rectangle region $[-2, 2] \times [-2, 2]$.

For the implementation of rebinning reconstruction algorithm (11), it is required to sample the variables Φ and α at a same grid distance. If θ is still sampled at this grid distance, we can avoid the interpolation procedure before the convolution because, for a fixed discrete pair (θ, α), $P_\Phi(\alpha)$ is exactly a sampled projection datum. On such discrete grids we accurately computed the line integrals of the phantom. Then we used these computer-generated projection data to reconstruct the phantom by algorithm (11).

Three focal length functions have been investigated:

1) $D(\alpha) = 2(1 + \tan^2 \alpha)$;
2) $D(\alpha) = 2/\cos \alpha$;
3) $D(\alpha) = 3 + 0.1|\sin \alpha|$.

From 1) to 3), $D'(\alpha)$ goes to smaller. In these simulations, Φ was evenly sampled by 128 points in $[0, 2\pi]$ and α was evenly sampled by 65 points on $[-(\pi/2), \pi/2]$. The grid distance of Φ and α is then the same. To code algorithm (11) we used C^{++} language. The program was executed first in a 486-66 DX2 PC. The total computation time for every focal length function was five minutes. By a Pentium II 266 PC, the computation time was reduced to less than half one minute. The reconstructed images are shown in Figs. 4–6 by 200×200 digitization. These numerical simulation procedures are almost the same as that for the parallel-beam geometry, so the numerical calculations are efficient. It is noted that when the focal length function $D(\alpha)$ changes from 1) to 3) gradually, the quality of reconstructed image becomes better successively. This phenomenon is due to the fact that a rapidly varying focal length function corresponds to a coarse grid in the parallel-beam geometry when the grids interval for α and for Φ are the same. In fact, the assumption that grid interval of Φ and α is the same does not satisfy the optimal relation of the sampling condition in [9, Ch. 3]. So, in the presented simulations, the effective grids are relatively small for collimators with rapidly varying focal length and there are more redundant data. In order to overcome these problems, a remedy is to adjust the grids interval of Φ and α. But this will need extra interpolations before convolution and may lead to nonuniformity artifacts. This interpolation problem can be solved by the use of the circular harmonic transform in the projection space, which is considered in [10].

VII. Discussion

Based on the angular representation of detector coordinates, we have proved that it is impossible to derive a convolution

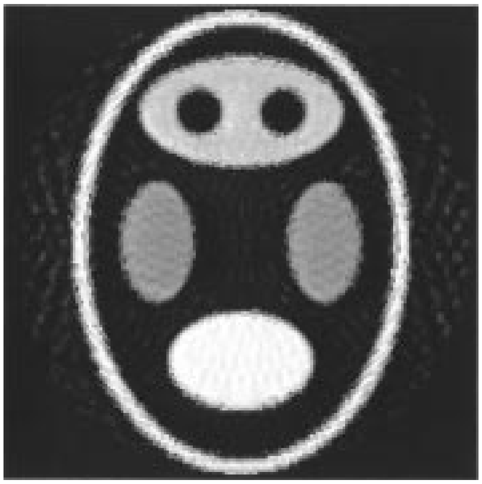

Fig. 6. Reconstructed image for focal length function 3).

backprojection algorithm for spatially varying focal length fan-beam geometry. So we turned to investigate some alternative efficient algorithms for the reconstruction problem of this type geometry. Algorithm (11) is very similar to a convolution backprojection procedure. In fact, it is equivalent to a convolution backprojection algorithm after rebinning the projection data for a specially chosen grids.

Algorithm (27) is a simplified form of the backprojection filtering algorithm [7] by the angular representation of detector coordinates. In the flat coordinate representation, it needs to solve the implicit equation (20). In general, (20) may not have a closed form solution for $\hat{s}$. However, when adopting the angular coordinate representation, this problem can be circumvented. For any spatially varying focal length function $D(\alpha)$, there always exists an explicit formula for (21). It is expected that an explicit formula may reduce the intermediate errors and the computation time resulting from calculating $\hat{s}$.

Let the implementation of (11) be Algorithm I and the implementation of (27) be Algorithm II. Interpolation is required in both algorithms. For instance, in Algorithm I, we make a strong assumption that the variables Φ, α and θ are sampled at a same grid distance, then we can avoid the rebinning interpolation before calculating the convolutions. But interpolation is needed in the coordinate transform from the polar to the Cartesian. In Algorithm II, interpolation is used before calculating the backprojection because $P_{\Phi-\alpha-\pi/2+\arccos[D(\alpha)\sin\alpha/r]}(\alpha)$ is not always the sampled projection datum for the discrete (x, y), Φ and α. It is noted that the interpolation in Algorithm I is taken after the numerical integral, while the interpolation in Algorithm II is taken before the backprojection between two nonparallel line integrals. So the interpolation in Algorithm II may possibly generate more intermediate errors and lead to nonuniformity artifacts in the reconstructed image.

When the grid distance of Φ is different from that of α in Algorithm I, we still need to interpolate the original data before calculating the convolution between two nonparallel line integrals. In general, such interpolation may degrade the quality of reconstructions and lead to nonuniformity artifacts in the reconstructed images. To overcome this problem, we have proposed a harmonic decomposition reconstruction algorithm in another paper [10].

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers for their many constructive suggestions and helpful comments on this paper. The authors also would like to express deep thanks to Dr. G. L. Zeng for reading this paper carefully and giving some critical comments.

REFERENCES

[1] L. A. Shepp and B. F. Logan, "The Fourier reconstruction of a head section," *IEEE Trans. Nucl. Sci.*, vol. NS-21, pp. 21–43, 1974.
[2] G. T. Herman and A. Naparstek, "Fast image reconstruction based on a Radon inversion formula appropriate for rapidly collected data," *SIAM J. Appl. Math.*, vol. 33, pp. 511–533, 1977.
[3] B. K. Horn, "Fan-beam reconstruction algorithms," *Proc. IEEE*, vol. 67, pp. 1616–1623, 1979.
[4] G. T. Gullberg, C. R. Crawford, and B. M. W. Tsui, "Reconstruction algorithm for fan-beam with a displaced center-of-rotation," *IEEE Trans. Med. Imag.*, vol. MI-5, pp. 23–29, 1986.
[5] G. L. Zeng, G. T. Gullberg, R. J. Jaszczak, and J. Li, "Fan-beam reconstruction algorithm for a spatially varying focal length collimator," *IEEE Trans. Med. Imag.*, vol. 12, pp. 575–582, 1993.
[6] Z. Cao and B. M. W. Tsui, "An analytical reconstruction algorithm for multifocal converging-beam SPECT," *Phys. Med. Biol.*, vol. 39, pp. 281–291, 1994.
[7] G. L. Zeng and G. T. Gullberg, "A backprojection filtering algorithm for a spatially varying focal length collimator," *IEEE Trans. Med. Imag.*, vol. MI-13, pp. 549–556, 1994.
[8] G. T. Herman, Ed., "Image reconstruction from projections," in *Topics in Applied Physics.* New York: Springer-Verlag, 1980, vol. 32.
[9] F. Natterer, *The Mathematics of Computerized Tomography.* New York: Wiley, 1986.
[10] J. You, "A harmonic decomposition reconstruction algorithm for spatially varying focal length fan-beam collimators," *IEEE Trans. Med. Imag.*, submitted for publication.